MATH 579 S26, Half-Exam 3 Solutions

1. Compute, with proof, $\Delta(n^{\overline{a}} + n^{\underline{a}})$, for $a \in \mathbb{N}_0$.
   Since Δ is a linear operator, we can compute each part separately, then add.
   We have $\Delta n^{\overline{a}} = (n+1)^{\overline{a}} - n^{\overline{a}} = (n+1)(n+2)\cdots(n+1+a-2)(n+1+a-1) - n(n+1)(n+2)\cdots(n+a-1) = (n+a-n)[(n+1)(n+2)\cdots(n+a-1)] = a[(n+1)(n+2)\cdots(n+a-1)] = a(n+1)^{\overline{a-1}}$.
   We have $\Delta n^{\underline{a}} = (n+1)^{\underline{a}} - n^{\underline{a}} = (n+1)(n)\cdots(n+1-a+1) - n(n-1)\cdots(n-a+2)(n-a+1) = ((n+1)-(n-a+1))[(n)\cdots(n-a+2)] = a[(n)\cdots(n-a+2)] = an^{\underline{a-1}}$.
   Combining, the answer is $\Delta(n^{\overline{a}} + n^{\underline{a}}) = a(n+1)^{\overline{a-1}} + an^{\underline{a-1}}$.

2. Prove the quotient rule $\Delta\frac{F(n)}{G(n)} = \frac{G(n)\Delta F(n) - F(n)\Delta G(n)}{G(n)G(n+1)}$.
   We calculate $\Delta\frac{F(n)}{G(n)} = \frac{F(n+1)}{G(n+1)} - \frac{F(n)}{G(n)} = \frac{F(n+1)G(n)-F(n)G(n+1)}{G(n+1)G(n)} = \frac{F(n+1)G(n)-F(n)G(n)+F(n)G(n)-F(n)G(n+1)}{G(n+1)G(n)}$, where we add and subtract $F(n)G(n)$ in the numerator. Now, we note that $F(n+1)G(n) - F(n)G(n) = (\Delta F(n))G(n)$ and $F(n)G(n) - F(n)G(n+1) = -(-F(n)G(n) + F(n)G(n+1)) = -F(n)\Delta G(n)$, so the desired result follows.

3. Use the FTDC to compute $\sum_{i=10}^{20} \binom{i}{4}$.
   We will first prove that $\Delta\binom{i}{5} = \binom{i}{4}$ (found via side calculation, or inspired guess). We calculate $\Delta\binom{i}{5} = \Delta\frac{i^{\underline{5}}}{5!} = (\frac{1}{5!})\Delta i^{\underline{5}} = \frac{1}{5!}5i^{\underline{4}} = \frac{i^{\underline{4}}}{4!} = \binom{i}{4}$.
   Now we rewrite to use difference calculus, noting that $10 \le i < 21$: $\sum_{i=10}^{20}\binom{i}{4} = \sum_{10}^{21}\binom{i}{4}\delta i$. Applying FTDC we get $\binom{i}{5}\Big|_{10}^{21} = \binom{21}{5} - \binom{10}{5} = 20349 - 252 = 20097$.

4. Use the FTDC to find, with proof, a formula for the sum of the first k fifth powers. Simplify for 1 point of extra credit.
   We seek $\sum_{i=1}^{k} i^5 = \sum_1^{k+1} i^5\delta i$. Now we apply exercise 1.25, which says that $i^5 = \sum_{j=0}^{5}\left\{{5 \atop j}\right\}i^{\underline{j}} = \left\{{5 \atop 0}\right\}i^{\underline{0}} + \left\{{5 \atop 1}\right\}i^{\underline{1}} + \left\{{5 \atop 2}\right\}i^{\underline{2}} + \left\{{5 \atop 3}\right\}i^{\underline{3}} + \left\{{5 \atop 4}\right\}i^{\underline{4}} + \left\{{5 \atop 5}\right\}i^{\underline{5}} = 0i^{\underline{0}} + 1i^{\underline{1}} + 15i^{\underline{2}} + 25i^{\underline{3}} + 10i^{\underline{4}} + 1i^{\underline{5}}$ (using the table of Stirling numbers from the back).
   Hence $\sum_1^{k+1} i^5\delta i = \sum_1^{k+1} 1i^{\underline{1}} + 15i^{\underline{2}} + 25i^{\underline{3}} + 10i^{\underline{4}} + 1i^{\underline{5}}\delta i = \frac{1}{2}i^{\underline{2}} + 5i^{\underline{3}} + \frac{25}{4}i^{\underline{4}} + 2i^{\underline{5}} + \frac{1}{6}i^{\underline{6}}\Big|_1^{k+1}$.
   Now, $\frac{1}{2}1^{\underline{2}} + 5\cdot 1^{\underline{3}} + \frac{25}{4}1^{\underline{4}} + 2\cdot 1^{\underline{5}} + \frac{1}{6}1^{\underline{6}} = 0+0+0+0+0 = 0$, so in fact this becomes $\frac{1}{2}(k+1)^{\underline{2}} + 5(k+1)^{\underline{3}} + \frac{25}{4}(k+1)^{\underline{4}} + 2(k+1)^{\underline{5}} + \frac{1}{6}(k+1)^{\underline{6}}$.

   You can simplify this as $\frac{4k^6+12k^5+10k^4-2k^2}{24}$ or even $\frac{k^2(k+1)^2(2k^2+2k-1)}{12}$.

   Fun fact: in the 17th century, Johann Faulhaber proved this equals $\frac{4a^3-a^2}{3}$, where $a = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. Further, he proved that if you replace the exponent 5 with any odd exponent $2m+1$, the sum $\sum_{i=1}^{k} i^{2m+1}$ is a nice function of this same a.

5. Use the FTDC to compute $\sum_{n=5}^{\infty}\frac{1}{(n-2)(n-1)n(n+1)}$.
   We seek $\sum_{n=5}^{N}\frac{1}{(n-2)(n-1)n(n+1)}$, to then take the limit as $N \to \infty$. Now, $\frac{1}{(n-2)(n-1)n(n+1)} = (n-3)^{\underline{-4}}$, which has anti-difference $\frac{-1}{3}(n-3)^{\underline{-3}}$. Applying the FTDC, we get $\frac{-1}{3}(n-3)^{\underline{-3}}\Big|_5^{N+1} = \frac{-1}{3}(N-2)^{\underline{-3}} + \frac{1}{3}(2)^{\underline{-3}}$. As $N \to \infty$, the first term goes to zero, leaving $\frac{1}{3}(2)^{\underline{-3}} = \frac{1}{3}\frac{1}{(3)(4)(5)} = \frac{1}{180}$.

6. Use summation by parts to compute $\sum_{n=0}^{20} n^{\overline{2}} n^{\underline{3}}$.

We seek $\sum_0^{21} n^{\overline{2}} n^{\underline{3}} \delta n$. Taking $F(n) = n^{\overline{2}}$ and $\Delta G(n) = n^{\underline{3}}$, we compute $\Delta F(n) = 2(n+1)^{\overline{1}} = 2(n+1)$ and $G(n) = \frac{1}{4} n^{\underline{4}}$. Using summation by parts we get $\sum_0^{21} n^{\overline{2}} n^{\underline{3}} \delta n = n^{\overline{2}} \left(\frac{1}{4} n^{\underline{4}}\right) \Big|_0^{21} - \sum_0^{21} \frac{1}{4}(n+1)^{\underline{4}} 2(n+1) \delta n = \frac{21^{\overline{2}} 21^{\underline{4}}}{4} - \frac{1}{2} \sum_0^{21} (n+1)^{\underline{4}} (n+1) \delta n = \frac{21^{\overline{2}} 21^{\underline{4}}}{4} - \frac{1}{2} \sum_1^{22} n^{\underline{4}} (n) \delta n$.
Turning now to $\sum_1^{22} n^{\underline{4}} (n) \delta n$, we take $F(n) = n$ and $\Delta G(n) = n^{\underline{4}}$ to do summation by parts again. We have $\Delta F(n) = 1$ and $G(n) = \frac{1}{5} n^{\underline{5}}$, so $\sum_1^{22} n^{\underline{4}} (n) \delta n = n \frac{n^{\underline{5}}}{5} \Big|_1^{22} - \sum_1^{22} 1 \frac{(n+1)^{\underline{5}}}{5} \delta n = \frac{22(22^{\underline{5}})}{5} - 0 - \frac{1}{30}(n+1)^{\underline{6}} \Big|_1^{22} = \frac{22(22^{\underline{5}})}{5} - \frac{23^{\underline{6}}}{30} - 0$.
Putting it all together, we get $\frac{21^{\overline{2}} 21^{\underline{4}}}{4} - \frac{1}{2} \left(\frac{22(22^{\underline{5}})}{5} - \frac{23^{\underline{6}}}{30} \right)$.

This happens to equal $10,849,608$, but you're not expected to calculate that without any tools.